

FLAT δ -VECTORS AND THEIR EHRHART POLYNOMIALS

TAKAYUKI HIBI AND AKIYOSHI TSUCHIYA

ABSTRACT. We call the δ -vector of an integral convex polytope of dimension d flat if the δ -vector is of the form $(1, 0, \dots, 0, a, \dots, a, 0, \dots, 0)$, where $a \geq 1$. In this paper, we give the complete characterization of possible flat δ -vectors. Moreover, for an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension d , we let $i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|$ and $i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$. By this characterization, we show that for any $d \geq 1$ and for any $k, \ell \geq 0$ with $k + \ell \leq d - 1$, there exist integral convex polytopes \mathcal{P} and \mathcal{Q} of dimension d such that (i) For $t = 1, \dots, k$, we have $i(\mathcal{P}, t) = i(\mathcal{Q}, t)$, (ii) For $t = 1, \dots, \ell$, we have $i^*(\mathcal{P}, t) = i^*(\mathcal{Q}, t)$ and (iii) $i(\mathcal{P}, k + 1) \neq i(\mathcal{Q}, k + 1)$ and $i^*(\mathcal{P}, \ell + 1) \neq i^*(\mathcal{Q}, \ell + 1)$.

INTRODUCTION

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\partial\mathcal{P}$ its boundary. Here an integral convex polytope is a convex polytope all of whose vertices have integer coordinates. For $n = 1, 2, \dots$, we write

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \quad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|,$$

where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ is the cardinality of a finite set X . The enumerative function $i(\mathcal{P}, n)$ has the following fundamental properties, which were studied originally in the work of Ehrhart [2]:

- $i(\mathcal{P}, n)$ is a polynomial in n of degree d ;
- $i(\mathcal{P}, 0) = 1$;
- (loi de r  ciprocit  ) $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

This polynomial $i(\mathcal{P}, n)$ is called the *Ehrhart polynomial* of \mathcal{P} . Consult [3, Part II] and [7, pp. 235–241] for fundamental materials on Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \dots$ of integers by the formula

$$(1 - \lambda)^{d+1} \sum_{n=0}^{\infty} i(\mathcal{P}, n) \lambda^n = \sum_{j=0}^{\infty} \delta_j \lambda^j.$$

Since $i(\mathcal{P}, n)$ is a polynomial in n of degree d , a fundamental fact on generating functions ([7, Corollary 4.3.1]) guarantees that $\delta_j = 0$ for every $j > d$. The sequence $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ is called the δ -vector of \mathcal{P} . The following properties on δ -vectors are well known:

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- $\delta_0 = 1, \delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$ and $\delta_d = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$;
- $i(\mathcal{P}, n) = \sum_{j=0}^d \binom{n+d-j}{d} \delta_j$;
- Each δ_i is nonnegative ([6]);
- Let $s = \max\{i : \delta_i \neq 0\}$. Then for $t = 1, \dots, d-s$, we have $i^*(\mathcal{P}, t) = 0$ and $i^*(\mathcal{P}, d-s+1) = \delta_s$;
- If $N = d$, the leading coefficient $(\sum_{i=0}^d \delta_i)/d!$ of $i(\mathcal{P}, n)$ is equal to the usual volume of \mathcal{P} ([7, Proposition 4.6.30]). In general, the positive integer $\text{vol}(\mathcal{P}) = \sum_{i=0}^d \delta_i$ is said to be the *normalized volume* of \mathcal{P} .

Through this paper, we assume that $N = d$.

We call the δ -vector of an integral convex polytope of dimension d *flat* if the δ -vector is of the form $(1, 0, \dots, 0, a, \dots, a, 0, \dots, 0)$, where $a \geq 1$. In this paper, we will give the complete characterization of possible flat δ -vectors. In fact, we show the following theorem.

Theorem 0.1. *Let $d \geq 1$ and $k, \ell \geq 0$ with $k + \ell \leq d - 1$ and $a \geq 1$. Given a finite sequence*

$$(\delta_0, \dots, \delta_d) = (1, \underbrace{0, \dots, 0}_k, a, \dots, a, \underbrace{0, \dots, 0}_\ell),$$

there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ -vector coincides with $(\delta_0, \dots, \delta_d)$ if and only if $k \leq \ell$.

This Theorem is a generalization of [5, Theorem 2.1].

Moreover, we consider the Ehrhart polynomials of flat δ -vectors. Let k and ℓ be positive integers, and let \mathcal{P} and \mathcal{Q} be integral convex polytopes of dimension d such that the following conditions are satisfied:

- For $t = 1, \dots, k$, we have $i(\mathcal{P}, t) = i(\mathcal{Q}, t)$;
- For $t = 1, \dots, \ell$, we have $i^*(\mathcal{P}, t) = i^*(\mathcal{Q}, t)$.

Since the degree of Ehrhart polynomials equal the dimension of underlying integral convex polytopes and the constant equals 1, and by Ehrhart reciprocity, if $k + \ell \geq d$, then we know that \mathcal{P} and \mathcal{Q} have a common Ehrhart polynomial. However, if $k + \ell \leq d - 1$, then \mathcal{P} and \mathcal{Q} don't necessarily have a common Ehrhart polynomial. By the characterization of flat δ -vectors, we will show the following theorems.

Theorem 0.2. *Let $d \geq 1$. Then for any $k, \ell \geq 0$ with $k + \ell \leq d - 1$, there exist integral convex polytopes \mathcal{P} and \mathcal{Q} of dimension d such that the followings are satisfied:*

- For $t = 1, \dots, k$, we have $i(\mathcal{P}, t) = i(\mathcal{Q}, t)$;
- For $t = 1, \dots, \ell$, we have $i^*(\mathcal{P}, t) = i^*(\mathcal{Q}, t)$;
- $i(\mathcal{P}, k+1) \neq i(\mathcal{Q}, k+1)$ and $i^*(\mathcal{P}, \ell+1) \neq i^*(\mathcal{Q}, \ell+1)$.

Theorem 0.3. *Let $d \geq 1$. Then for any $0 \leq k \leq \ell \leq d - k - 1$, there exists an infinite family $\{\mathcal{P}_1, \mathcal{P}_2, \dots\}$ of integral convex polytopes of dimension d such that for each \mathcal{P}_i and \mathcal{P}_j with $i \neq j$, the followings are satisfied:*

- *For $t = 1, \dots, k$, we have $i(\mathcal{P}_i, t) = i(\mathcal{P}_j, t)$;*
- *For $t = 1, \dots, \ell$, we have $i^*(\mathcal{P}_i, t) = i^*(\mathcal{P}_j, t)$;*
- *$i(\mathcal{P}_i, k + 1) \neq i(\mathcal{P}_j, k + 1)$ and $i^*(\mathcal{P}_i, \ell + 1) \neq i^*(\mathcal{P}_j, \ell + 1)$.*

In Section 1, we recall the some properties and the calculation method on the δ -vectors of integral simplices. In Section 2, we prove Theorem 0.1, 0.2 and 0.3.

1. PRELIMINARIES

At first, we recall some properties of δ -vectors. There are two well-known inequalities on δ -vectors. Let $s = \max \{i : \delta_i \neq 0\}$. One inequality is

$$(1) \quad \delta_0 + \delta_1 + \dots + \delta_i \leq \delta_s + \delta_{s-1} + \dots + \delta_{s-i}, \quad 0 \leq i \leq \lfloor s/2 \rfloor,$$

which was proved by Stanley [8], and another one is

$$(2) \quad \delta_d + \delta_{d-1} + \dots + \delta_{d-i} \leq \delta_1 + \delta_2 + \dots + \delta_{i+1}, \quad 0 \leq i \leq \lfloor (d-1)/2 \rfloor,$$

which appears in the work of the first author [4, Remark 1.4]. Also, there are more recent and more general results on δ -vectors by Alan Stapledon in [9].

Moreover, we recall the following lemma.

Lemma 1.1. *Suppose that $(\delta_0, \delta_1, \dots, \delta_d)$ is the δ -vector of an integral convex polytope of dimension d . Then there exists an integral convex polytope of dimension $d+1$ whose δ -vector is $(\delta_0, \delta_1, \dots, \delta_d, 0)$.*

Next, we recall the well-known combinatorial technique how to compute the δ -vector of an integral simplex. Given an integral simplex $\mathcal{F} \subset \mathbb{R}^d$ of dimension d with the vertices $v_0, v_1, \dots, v_d \in \mathbb{R}^d$, we set

$$\text{Box}(\mathcal{F}) = \left\{ \alpha \in \mathbb{Z}^{d+1} : \alpha = \sum_{i=0}^d \lambda_i(v_i, 1), \quad 0 \leq \lambda_i < 1 \right\}.$$

We define the degree of $\alpha = \sum_{i=0}^d \lambda_i(v_i, 1) \in \text{Box}(\mathcal{F}) \cap \mathbb{Z}^d$ with $\deg(\alpha) = \sum_{i=0}^d \lambda_i$, i.e., the last coordinate of α . Then we have the following lemma.

Lemma 1.2. *Let $\delta(\mathcal{F}) = (\delta_0, \delta_1, \dots, \delta_d)$. Then each δ_i is equal to the number of integer points $\alpha \in \text{Box}(\mathcal{F})$ with $\deg(\alpha) = i$.*

In particular, if $v_0 = (0, 0, \dots, 0)$, then for $\alpha = \sum_{i=0}^d \lambda_i(v_i, 1) \in \text{Box}(\mathcal{F}) \cap \mathbb{Z}^d$, we have $\deg(\alpha) = \left\lceil \sum_{i=1}^d \lambda_i \right\rceil$.

2. PROOFS OF THEOREMS

At first, in order to prove Theorem 0.1, we show the following lemmas.

Lemma 2.1. *Let $d \geq 3$. For any $1 \leq k \leq \lfloor (d-1)/2 \rfloor$ and for any $a \geq 1$, there exists an integral convex polytope \mathcal{P} of dimension d such that*

$$\delta(\mathcal{P}) = (1, \underbrace{0, \dots, 0}_k, a, \dots, a, \underbrace{0, \dots, 0}_k).$$

Proof. We set $\mathcal{P} = \text{conv}(\{v_0, \dots, v_d\}) \subset \mathbb{R}^d$, where

$$v_i = \begin{cases} (0, \dots, 0), & \text{if } i = 0, \\ e_i, & \text{if } 1 \leq i \leq d-1, \\ \sum_{j=1}^{d-k} e_j + a(d-2k) \sum_{j=d-k+1}^{d-1} e_j + (a(d-2k)+1)e_d, & \text{if } i = d, \end{cases}$$

where e_1, \dots, e_d are the canonical unit coordinate vectors of \mathbb{R}^d . We compute the δ -vector of \mathcal{P} . Let $\lambda_1, \dots, \lambda_d \in [0, 1)$ such that $\sum_{i=1}^d \lambda_i v_i \in \mathbb{Z}^d$. Then there exists an integer t with $1 \leq t \leq a(d-2k)$ such that $\lambda_d = \frac{t}{a(d-2k)+1}$. Hence we have

$$\lambda_i = \begin{cases} \frac{a(d-2k)+1-t}{a(d-2k)+1}, & \text{if } 1 \leq i \leq d-k, \\ \frac{t}{a(d-2k)+1}, & \text{if } d-k+1 \leq i \leq d-1. \end{cases}$$

For $1 \leq t \leq a(d-2k)$, we let $f(t) = \frac{a(d-2k)+1-t}{a(d-2k)+1}(d-k) + \frac{kt}{a(d-2k)+1}$. Then we have $f(t) = d-k - \frac{t(d-2k)}{a(d-2k)+1}$. Let $0 \leq \ell \leq d-2k-1$ and $1 \leq j \leq a$. Since

$$\begin{aligned} f(a\ell+j) &= d-k - \frac{(a\ell+j)(d-2k)}{a(d-2k)+1} \\ &= d-k - \frac{\ell(a(d-2k)+1) - \ell + j(d-2k)}{a(d-2k)+1} \\ &= d-k - \ell - \frac{j(d-2k) - \ell}{a(d-2k)+1}, \end{aligned}$$

we have $\lceil f(a\ell+j) \rceil = d-k-\ell$. Hence we know that

$$\delta_i = \begin{cases} 1, & \text{if } i = 0, \\ a, & \text{if } k+1 \leq i \leq d-k, \\ 0, & \text{otherwise,} \end{cases}$$

as desired. \square

Lemma 2.2. *For any $d \geq 1$ and for any $a \geq 1$, there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ such that $\delta(\mathcal{P}) = (1, a, \dots, a)$.*

Proof. We set $\mathcal{P} = \text{conv}(\{v_0, \dots, v_d\}) \subset \mathbb{R}^d$, where

$$v_i = \begin{cases} (0, \dots, 0), & \text{if } i = 0, \\ e_i, & \text{if } 1 \leq i \leq d-1, \\ ad \sum_{j=1}^{d-1} e_j + (ad+1)e_d, & \text{if } i = d. \end{cases}$$

We compute the δ -vector of \mathcal{P} . Let $\lambda_1, \dots, \lambda_d \in [0, 1)$ such that $\sum_{i=1}^d \lambda_i v_i \in \mathbb{Z}^d$. Then there exists an integer t with $1 \leq t \leq ad$ such that $\lambda_d = \frac{t}{ad+1}$. Hence for $1 \leq i \leq d-1$, we have $\lambda_i = \frac{t}{ad+1}$. For $1 \leq t \leq ad$, we let $f(t) = \frac{dt}{ad+1}$, $0 \leq k \leq d-1$ and $1 \leq j \leq a$. Since

$$f(ka+j) = \frac{d(ka+j)}{ad+1} = \frac{k(ad+1) - k + dj}{ad+1} = k + \frac{jd-k}{ad+1},$$

we have $\lceil f(ka+j) \rceil = k+1$. Hence we know that

$$\delta_i = \begin{cases} 1, & \text{if } i = 0, \\ a, & \text{if } 1 \leq i \leq d. \end{cases}$$

as desired. \square

By using Lemma 1.1, 2.1 and 2.2, and Hibi's inequality, the assertion of Theorem 0.1 follows.

Next, we prove Theorem 0.2.

Proof of Theorem 0.2. By Theorem 0.1, there exist integral convex polytopes \mathcal{P} and \mathcal{Q} of dimension d such that

$$\delta_i(\mathcal{P}) = \begin{cases} 1, & \text{if } i = 0, \\ a, & \text{if } 1 \leq i \leq k, \\ 0, & \text{if otherwise,} \end{cases}$$

and

$$\delta_i(\mathcal{Q}) = \begin{cases} 1, & \text{if } i = 0, \\ a, & \text{if } 1 \leq i \leq d-\ell, \\ 0, & \text{if otherwise,} \end{cases}$$

where $a \geq 1$. Then we know that for $t = 1, \dots, \ell$, $i^*(\mathcal{P}, t) = i^*(\mathcal{Q}, t) = 0$ and $0 = i^*(\mathcal{P}, \ell + 1) \neq i^*(\mathcal{Q}, \ell + 1) = a$. Since

$$i(\mathcal{P}, n) = \binom{n+d}{d} + a \sum_{i=1}^k \binom{n+d-i}{d}$$

and

$$i(\mathcal{Q}, n) = \binom{n+d}{d} + a \sum_{i=1}^{d-\ell} \binom{n+d-i}{d}$$

and since for $k+1 \leq i \leq d-\ell$ and for $1 \leq t \leq k$, we have $\binom{t+d-i}{d} = 0$, we know that for $t = 1, \dots, k$, we have $i(\mathcal{P}, t) = i(\mathcal{Q}, t)$. Moreover, since $\sum_{i=k+1}^{d-\ell} \binom{k+1+d-i}{d} = 1$, we have $i(\mathcal{P}, k+1) \neq i(\mathcal{Q}, k+1)$, as desired. \square

Next, we prove Theorem 0.3.

Proof of Theorem 0.3. By Proposition 0.1, there exist integral convex polytopes \mathcal{P} and \mathcal{Q} of dimension d such that

$$\delta_i(\mathcal{P}) = \begin{cases} 1, & \text{if } i = 0, \\ a, & \text{if } k+1 \leq i \leq d-\ell, \\ 0, & \text{if otherwise,} \end{cases}$$

and

$$\delta_i(\mathcal{Q}) = \begin{cases} 1, & \text{if } i = 0, \\ b, & \text{if } k+1 \leq i \leq d-\ell, \\ 0, & \text{if otherwise,} \end{cases}$$

where $1 \leq a < b$. Then we know that for $t = 1, \dots, \ell$, $i^*(\mathcal{P}, t) = i^*(\mathcal{Q}, t) = 0$ and $a = i^*(\mathcal{P}, \ell + 1) \neq i^*(\mathcal{Q}, \ell + 1) = b$. Since

$$i(\mathcal{P}, n) = \binom{n+d}{d} + a \sum_{i=k+1}^{d-\ell} \binom{n+d-i}{d}$$

and

$$i(\mathcal{Q}, n) = \binom{n+d}{d} + b \sum_{i=k+1}^{d-\ell} \binom{n+d-i}{d}$$

and since for $k+1 \leq i \leq d-\ell$ and for $1 \leq t \leq k$, we have $\binom{t+d-i}{d} = 0$, we know that for $t = 1, \dots, k$, we have $i(\mathcal{P}, t) = i(\mathcal{Q}, t)$. Moreover, since $\sum_{i=k+1}^{d-\ell} \binom{k+1+d-i}{d} = 1$, we have $i(\mathcal{P}, k+1) \neq i(\mathcal{Q}, k+1)$. \square

By using integral convex polytopes with flat δ -vectors, we can construct an infinite family in Theorem 0.3. However, for $0 \leq \ell < k \leq d-\ell-1$, we cannot construct such an infinite family. Finally, we give the following question.

Question 2.3. *Let $d \geq 1$. Then for any $0 \leq \ell < k \leq d - \ell - 1$, does there exist an infinite family $\{\mathcal{P}_1, \mathcal{P}_2, \dots\}$ of integral convex polytopes of dimension d such that for each \mathcal{P}_i and \mathcal{P}_j with $i \neq j$, the followings are satisfied:*

- *For $t = 1, \dots, k$, we have $i(\mathcal{P}_i, t) = i(\mathcal{P}_j, t)$;*
- *For $t = 1, \dots, \ell$, we have $i^*(\mathcal{P}_i, t) = i^*(\mathcal{P}_j, t)$;*
- *$i(\mathcal{P}_i, k + 1) \neq i(\mathcal{P}_j, k + 1)$ and $i^*(\mathcal{P}_i, \ell + 1) \neq i^*(\mathcal{P}_j, \ell + 1)$?*

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(Takayuki Hibi) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN

E-mail address: hibi@math.sci.osaka-u.ac.jp

(Akiyoshi Tsuchiya) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN

E-mail address: a-tsuchiya@cr.math.sci.osaka-u.ac.jp